Fundamental matrix solution of $\dot{x} = Ax$

Q). Define the fundamental matrix solution of $\dot{x} = Ax$
where $x = [x_1, x_2, \dots x_n]^T$
A is a matrix of order n.

The given system of diff. eq^n is

$$\dot{x} = Ax \quad \text{——— (1)}$$

let

$$x^{(1)} = x^{(1)}(t)$$
$$x^{(2)} = x^{(2)}(t)$$
$$- - - - - - ,$$
$$- - - - - - ,$$
$$x^{(n)} = x^{(n)}(t)$$

are n linearly independent sol^n of (1)

Then

$$X(t) = [x^{(1)} \; x^{(2)} \dots x^{(n)}]$$

is called fundamental matrix sol^n of eq^n(1).

— A matrix $X(t)$ is called a fundamental matrix sol^n of (1) if its columns form a set of n linearly independent sol^n of (1).

Q) Show that $e^{At} = X(t)\,X^{-1}(0)$
where, $X(t)$ is the fundamental matrix solution of system of ordinary diff. eq^n $\dot{x} = Ax$

Sol^n

Assume that the system of O.D.E. is

$$\dot{x} = Ax \quad \text{——} \quad (1)$$

where $x = [x_1, x_2, \ldots x_n]^T$
and A is $n \times n$ matrix with real constant entries.

If $x^{(1)}, x^{(2)} \ldots x^{(n)}$ are linearly independent solution of (1) then

$$X(t) = [x^{(1)}, x^{(2)} \ldots x^{(n)}]$$

is called fundamental matrix sol^n.

First we prove the following three lemma —

Lemma I :
$X(t)$ is a fundamental matrix sol^n of (1) iff $X(t)$ satisfies (1) and $|X(0)| \neq 0$

Lemma II:
e^{At} is a fundamental matrix solⁿ of ①

Lemma - III:
If $X(t)$ and $Y(t)$ are two fundamental matrix solution of ① then
$$Y(t) = X(t) \cdot C$$
where, C is an $n \times n$ matrix of real constants

Proof of lemma I

Let $X(t) = [x^{(1)}, x^{(2)}, \dots x^{(n)}]$
be a fundamental matrix solution of ①
Then $x^{(1)}, x^{(2)}, \dots x^{(n)}$ are linearly independent solⁿˢ of ①
i.e.
$$\dot{x}^{(1)} = A x^{(1)}$$
$$\dot{x}^{(2)} = A x^{(2)}$$
$$\dots$$
$$\dot{x}^{(n)} = A x^{(n)}$$

i.e. $[\dot{x}^{(1)}\ \dot{x}^{(2)} \dots \dot{x}^{(n)}] = A\,[x^{(1)}\, x^{(2)} \dots x^{(n)}]$.

ie $\dot{X}(t) = AX(t)$.

$\Rightarrow$ $X(t)$ satisfies (1) and Column of $X(0)$ are linearly Independent.

$\Rightarrow |X(0)| \neq 0$

Next, we assume that $X(t)$ satisfies (1) and $X(0) \neq 0$

we have

$$\dot{X}(t) = AX(t)$$

ie. $[\dot{x}^{(1)}\ \dot{x}^{(2)} \dots \dot{x}^{(n)}] = A\,[x^{(1)}\ x^{(2)} \dots x^{(n)}]$

$$\Rightarrow \quad \begin{aligned} \dot{x}^{(1)} &= A x^{(1)} \\ \dot{x}^{(2)} &= A x^{(2)} \\ &\dots\dots \\ &\dots\dots \\ \dot{x}^{(n)} &= A x^{(n)} \end{aligned}$$

ie. $\dot{x}^{(i)}$ satisfies (1) for $i = 1, 2, \dots n$

Now, we claim that columns of $X(t)$ are linearly independent.

for if we assume that
$x^{(1)}(t), x^{(2)}(t) \dots \dots x^{(n)}(t)$
are linearly dependent then they are linearly dependent for every value of t.
In particular $x^{(i)}(0)$ are linearly dependent
i.e. $|X(0)| = 0$
which is a contradiction.

Hence $X(t)$ is a fundamental matrix sol^n of ①

<u>Proof of lemma II</u>

we know that
$$\frac{d}{dt}(e^{At}) = Ae^{At}$$

Hence $X(t) = e^{At}$ satisfies ①

Also $|X(0)| = e^{0t} = |I| \neq 0$

Hence by lemma I
e^{At} is a fundamental matrix sol^n of ①

Proof of lemma III :

Let $Y(t) = [y^{(1)}, y^{(2)} \dots y^{(n)}]$ be another sol^n of (1)

Hence $\{y^{(j)}, j = 1, 2, \dots n\}$ and $\{x^{(i)}, i = 1, 2, \dots n\}$ form two bases of the solution space. Hence $y^{(j)}$ can be expressed as a linear combination of $x^{(i)}$ for $i = 1, 2, \dots n$, $j = 1, 2, \dots n$.

ie $$y^{(j)} = \sum_{i=1}^{n} c_{ij} \, x^{(i)}$$

$$y^{(j)} = c_{1j} x^{(1)} + c_{2j} x^{(2)} + \dots + c_{nj} x^{(n)}$$

$$y^{(j)} = \left[x^{(1)} \; x^{(2)} \dots x^{(n)} \right] \begin{bmatrix} c_{1j} \\ c_{2j} \\ \vdots \\ c_{nj} \end{bmatrix}$$

$$\left[y^{(1)} \, y^{(2)} \dots y^{(n)} \right] = \left[x^{(1)} \, x^{(2)} \dots x^{(n)} \right] \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ - & - & - & - \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{bmatrix}$$

i.e $Y(t) = X(t) \cdot C$

where $C = [c_{ij}]$, $i = 1, 2, \dots n$
$j = 1, 2, \dots n$

This proves lemma III

Proof of the theorem:

By lemma II
$Y(t) = e^{At}$ is a solⁿ of ①

By lemma III
$e^{At} = X(t) \cdot C$ —— ②

putting $t = 0$, we get
$e^{0} = X(0) \cdot C$
$I = X(0) C \Rightarrow C = X^{-1}(0)$

putting $C = X^{-1}(0)$ in ②, we get

$$\boxed{e^{At} = X(t) X^{-1}(0)}$$

Q). Find a fundamental matrix solution of the system of differential equations.

$$\dot{x} = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} x$$

Q) Find e^{At} if $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 5 \end{bmatrix}$

Q). Find e^{At} if $A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$